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ON THE STRICTION CURVES OF INVOLUTIVE FRENET RULED SURFACES IN $\mathbb{E}^3$

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Abstract. In this article we conceive eight ruled surfaces related to the evolute curve α and involute α^*. They are called as Frenet ruled surface and involutive Frenet ruled surfaces, cause of their generators are Frenet vector fields of evolute curve α. First we give tangent vector fields of striction curves of all Frenet ruled surfaces and the tangent vector fields of striction curves of involutive Frenet ruled surfaces are given according to Frenet apparatus of evolute curve α. Further we give only one matrix in which we can see sixteen position of these tangent vector fields, such that we can say there is six position the tangent vector fields are perpendicular.

1. General Information

Deriving curves based on the other curves is a subject in geometry. Bertrand curves, involute-evolute curves are this kind of curves. By using the analogous means we generate ruled surface based on the other ruled surface. The properties of the B-scroll are also examined in Euclidean 3-space, Lorentzian 3-space and n-space with time-like directrix curve and null rulings (see [2], [5], [6]). Differential geometric elements of the *involute* $\tilde{D}$ *scroll* are examined in [10]. Let Frenet vector fields be $V_1(s), V_2(s), V_3(s)$ of α and let first and second curvatures of the curve $\alpha(s)$ be $k_1(s)$ and $k_2(s)$, respectively. The quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are Frenet-Serret elements of the curves. Frenet formulae are,

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{1.1}$$

The Darboux vector makes a path of curvature k_1 and torsion k_2, curvature is the measuring of the rotation of the Frenet frame on the binormal unit vector, and torsion is the measurement of the rotation of the Frenet frame on the tangent unit

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vector. For any unit speed curve α, according to the Frenet-Serret elements, the Darboux vector can be defined

$$D(s) = k_2(s)V_1(s) + k_1(s)V_3(s) \tag{1.2}$$

where curvature functions are defined by $k_1(s) = \|V_1(s)\|$ and $k_2(s) = -\left\langle V_2, \dot{V}_3 \right\rangle$. The Darboux vector field of α and it has the bellowing symmetrical properties, [3].

$$\tilde{D}(s) = \frac{k_2}{k_1}(s)V_1(s) + V_3(s) \tag{1.3}$$

throughout $\alpha(s)$ under the condition that $k_1(s) \neq 0$ and it is called the modified Darboux vector field of α [8].
Let unit speed regular curve $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be given. For $\forall s \in I$, then the curve α^* is called the involute of the curve α, if the tangent at the point $\alpha(s)$ to the curve α passes through the tangent at the point $\alpha^*(s)$ to the curve α^*, then we can write that

$$\alpha^*(s) = \alpha(s) + (c - s)V_1(s), c = const.$$

The distance between corresponding points of the involute curve in $\mathbb{E}^3$ is $d\big(\alpha(s), \alpha^*(s)\big) = |c - s|$, c is constant ,$\forall s \in I$, ([4],[9]). The Frenet vector fields of the *involute* α^*, based on the its evolute curve α are

$$\begin{cases} V_1^* = V_2, \\ V_2^* = \frac{-k_1}{(k_1^2+k_2^2)^{\frac{1}{2}}}V_1 + \frac{k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}}V_3 \\ V_3^* = \frac{k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}}V_1 + \frac{k_1}{(k_1^2+k_2^2)^{\frac{1}{2}}}V_3 \end{cases} \tag{1.4}$$

and

$$\tilde{D}^* = \frac{k_2}{(k_1^2 + k_2^2)^{\frac{1}{2}}}V_1 - \frac{k_1' k_2 - k_1 k_2'}{(k_1^2 + k_2^2)^{\frac{3}{2}}}V_2 + \frac{k_1 V_3}{(k_1^2 + k_2^2)^{\frac{1}{2}}}. \tag{1.5}$$

The first curvature and second curvature of *involute* α^* are, respectively [9],

$$k_1^* = \frac{\sqrt{k_1^2 + k_2^2}}{(c - s)k_1}, \quad k_2^* = \frac{-k_2^2\left(\frac{k_1}{k_2}\right)'}{(c - s)k_1\,(k_1^2 + k_2^2)}. \tag{1.6}$$

Since $\eta = k_1^2 + k_2^2 \neq 0$, and $\mu = \left(\frac{k_2}{k_1}\right)'$, we have

$$\eta^* = k_1^{*2} + k_2^{*2} = \left(\frac{\sqrt{k_1^2 + k_2^2}}{\lambda k_1}\right)^2 + \left(\frac{k_2' k_1 - k_1' k_2}{\lambda k_1\,(k_1^2 + k_2^2)}\right)^2 = \frac{\eta^3 + k_1^4\mu^2}{\lambda^2\eta^2 k_1^2}, \tag{1.7}$$

$$\mu^* = \left(\frac{k_2^*}{k_1^*}\right)' \frac{ds}{ds^*} = \frac{\frac{k_2' k_1 - k_1' k_2}{\lambda k_1 (k_1^2+k_2^2)}}{\frac{\sqrt{k_1^2+k_2^2}}{\lambda k_1}} \frac{1}{\lambda k_1} = \frac{k_2' k_1 - k_1' k_2}{\lambda k_1\,(k_1^2 + k_2^2)^{\frac{3}{2}}} = \frac{\mu k_1}{\lambda \eta^{\frac{3}{2}}}, \tag{1.8}$$

$$\left(\frac{k_1^*}{\eta^*}\right)' = \left(\frac{\frac{\sqrt{k_1^2+k_2^2}}{\lambda k_1}}{\frac{(k_1^2+k_2^2)^3 + (k_2' k_1 - k_1' k_2)^2}{\lambda^2 k_1^2 (k_1^2+k_2^2)^2}}\right)' \frac{1}{\lambda k_1} = \left(\frac{\eta^{\frac{5}{2}}\lambda k_1}{\eta^3 + k_1^2\mu}\right)' \frac{1}{\lambda k_1}. \tag{1.9}$$

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation ,

$$\varphi(s,v)=\alpha(s)+vx(s), \tag{1.10}$$

where α and x are curves in $\mathbb{E}^3$. We call φ a ruled patch. The curve α is called the directrix or base curve of the ruled surface, and x is called the director curve, [1]. The striction point on a ruled surface is the foot of the common normal between two consecutive generators. The set of striction points defines the striction curve given by [1]

$$c(s)=\alpha(s)-\frac{\langle \alpha_s,x_s\rangle}{\langle x_s,x_s\rangle}x(s). \tag{1.11}$$

2. On the striction curves of Involutive Frenet ruled surfaces in $\mathbb{E}^3$

Theorem 2.1. *The striction curves of Frenet ruled surfaces are,* [7]

$$\begin{bmatrix} c_1-\alpha \\ c_2-\alpha \\ c_3-\alpha \\ c_4-\alpha \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{k_1}{k_2^2+k_2^2} & 0 \\ 0 & 0 & 0 \\ \frac{-k_2}{k_1\left(\frac{k_2}{k_1}\right)'} & 0 & \frac{-1}{\left(\frac{k_2}{k_1}\right)'} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{2.1}$$

Theorem 2.2. *Tangent vector fields* T_1, T_2, T_3, *and* T_4 *of striction curves along Frenet ruled surface are given by*

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ \frac{k_2^2}{\eta\|c_2'(s)\|} & \frac{\left(\frac{k_1}{\eta}\right)'}{\|c_2'(s)\|} & \frac{k_1k_2}{\eta\|c_2'(s)\|} \\ 1 & 0 & 0 \\ \frac{\mu-\mu'-\frac{k_2}{k_1}}{\mu\|c_4'(s)\|} & 0 & \frac{\mu'}{\mu^2\|c_4'(s)\|} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where $k_1^2+k_2^2=\eta$, $\left(\frac{k_2}{k_1}\right)'=\mu$.

Proof. It is given this matrix, so we get equalyties as follows:

$$T_1(s)=T_3(s)=\alpha'(s)=V_1$$

Since $c_2(s)=\alpha(s)+\frac{k_1}{k_1^2+k_2^2}V_2$ and

$$T_2(s)=\frac{k_2^2}{(k_1^2+k_2^2)\|c_2'(s)\|}V_1+\frac{\left(\frac{k_1}{\eta}\right)'}{(k_1^2+k_2^2)\|c_2'(s)\|}V_2+\frac{k_1k_2}{(k_1^2+k_2^2)\|c_2'(s)\|}V_3.$$

Also

$$\begin{aligned} T_4(s) &= \frac{\left(\left(\frac{k_2}{k_1}\right)'\right)^2-\left(\frac{k_2}{k_1}\right)'\left(\frac{k_2}{k_1}\right)''-\frac{k_2}{k_1}\left(\frac{k_2}{k_1}\right)'}{\left(\left(\frac{k_2}{k_1}\right)'\right)^2\|c_4'(s)\|}V_1-\frac{-1\left(\frac{k_2}{k_1}\right)''}{\left(\left(\frac{k_2}{k_1}\right)'\right)^2\|c_4'(s)\|}V_3, \\ T_4(s) &= \frac{\mu^2-\mu\mu'-\frac{k_2}{k_1}\mu}{\mu^2\|c_4'(s)\|}V_1+\frac{\mu'}{\mu^2\|c_4'(s)\|}V_3. \end{aligned}$$

□

Definition 2.1. Let $\alpha^*(s)$ be involute of $\alpha(s)$ with arc-lenght parameter s. The equations

$$\begin{cases} \varphi_1^*(s,v_1) = \alpha^*(s) + v_1 V_1^*(s) \\ \varphi_2^*(s,v_2) = \alpha^*(s) + v_2 V_2^*(s) \\ \varphi_3^*(s,v_3) = \alpha^*(s) + v_3 V_3^*(s) \\ \varphi_4^*(s,v_4) = \alpha^*(s) + v_4 \tilde{D}^*(s) \end{cases}$$

are the parametrization of Frenet ruled surface of involute curve $\alpha^*(s)$.

The above definition can be written as follows.

$$\begin{cases} \varphi_1^*(s,v_1) = \alpha(s) + (\sigma - s)V_1(s) + v_1 V_2(s), \\ \varphi_2^*(s,v_2) = \alpha(s) + (\sigma - s)V_1(s) + v_2\left(\frac{-k_1V_1+k_2V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\right), \\ \varphi_3^*(s,v_3) = \alpha(s) + (\sigma - s)V_1(s) + v_3\left(\frac{k_2V_1+k_1V_3}{(k_1^2+k_2^2)^{\frac{1}{2}}}\right), \\ \varphi_4^*(s,v_4) = \alpha(s) + (\sigma - s)V_1(s) \\ \qquad +v_4\left(\frac{k_2}{\sqrt{k_1^2+k_2^2}}V_1 - \frac{k_1'k_2-k_1k_2'}{(k_1^2+k_2^2)^{\frac{3}{2}}}V_2 + \frac{k_1V_3}{\sqrt{k_1^2+k_2^2}}\right) \end{cases}$$

Theorem 2.3. *The equations of the striction curves of involutive Frenet ruled surfaces on the evolute curve α according to Frenet elements of evolute curve α,* [7]

$$\begin{bmatrix} c_1^* - \alpha \\ c_2^* - \alpha \\ c_3^* - \alpha \\ c_4^* - \alpha \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ \lambda\left(1 - \frac{k_1^2}{\eta(1+m)}\right) & 0 & \lambda\frac{k_1k_2}{\eta(1+m)} \\ \lambda & 0 & 0 \\ \lambda - \frac{k_2}{m'\ \eta^{\frac{1}{2}}} & -\frac{m}{m'} & \frac{k_1}{m'\eta^{\frac{1}{2}}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{2.2}$$

Theorem 2.4. *Tangent vector fields $T_1{}^*, T_2{}^*, T_3{}^*, T_4{}^*$ of striction curves of involutive Frenet ruled surface according to Frenet elements by themselves are given by*

$$\begin{bmatrix} T_1{}^* \\ T_2{}^* \\ T_3{}^* \\ T_4{}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{-b^*k_1 + c^*k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}} & a^* & \dfrac{b^*k_2 + c^*k_1}{(k_1^2+k_2^2)^{\frac{1}{2}}} \\ 0 & 1 & 0 \\ \dfrac{e^*k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}} & d^* & \dfrac{e^*k_1}{(k_1^2+k_2^2)^{\frac{1}{2}}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{2.3}$$

where

$$\begin{aligned} a^* &= \frac{{k_2^*}^2}{\eta^*\left\|{c_2^*}'(s)\right\|}, \quad b^* = \frac{\left(\frac{k_1^*}{\eta^*}\right)'}{\left\|{c_2^*}'(s)\right\|}, \quad c^* = \frac{k_1^*k_2^*}{\eta^*\left\|{c_2^*}'(s)\right\|} \\ d^* &= \frac{\mu^* - {\mu^*}' - \frac{k_2^*}{k_1^*}}{\mu^*\left\|{c_4^*}'(s)\right\|}, \quad e^* = \frac{{\mu^*}'}{{\mu^*}^2\left\|{c_4^*}'(s)\right\|} \end{aligned}$$

and ${k_1^*}^2 + {k_2^*}^2 = \eta^*$, $\left(\frac{k_2^*}{k_1^*}\right)' = \mu^*$.

Proof. Tangent vector fields ${T_1}^*, {T_2}^*, {T_3}^*, {T_4}^*$ of striction curves of involutive Frenet ruled surface matrix form as follows;

$$\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \\ T_4^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d^* & 0 & e^* \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}.$$

In the above matrix by using the equation (1.2), we can write

$$\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \\ T_4^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d^* & 0 & e^* \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{-k_1}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} & 0 & \frac{k_2}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} \\ \frac{k_2}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} & 0 & \frac{k_1}{\left(k_1^2+k_2^2\right)^{\frac{1}{2}}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

or

$$\begin{bmatrix} {T_1}^* \\ {T_2}^* \\ {T_3}^* \\ {T_4}^* \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{-b^*k_1 + c^*k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}} & a^* & \dfrac{b^*k_2 + c^*k_1}{(k_1^2+k_2^2)^{\frac{1}{2}}} \\ 0 & 1 & 0 \\ \dfrac{e^*k_2}{(k_1^2+k_2^2)^{\frac{1}{2}}} & d^* & \dfrac{e^*k_1}{(k_1^2+k_2^2)^{\frac{1}{2}}} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

□

Theorem 2.5. *The product of tangent vector fields* $T_1^*,\ T_2^*,\ T_3^*,\ T_4^*$ *and tangent vector fields* $T_1,\ T_2,\ T_3,\ T_4,$ *of striction curves belonging to Frenet ruled surfaces and involutive Frenet ruled surfaces are given by,*

$$[T]\,[T^*]^{\mathbf{T}} = \frac{1}{\eta^{\frac{1}{2}}} \begin{bmatrix} 0 & -k_1b^* + k_2c^* & 0 & k_2e^* \\ b\eta^{\frac{1}{2}} & X & b\eta^{\frac{1}{2}} & b\eta^{\frac{1}{2}}d^* + (ak_2 + ck_1)\,e^* \\ 0 & -k_1b^* + k_2c^* & 0 & k_2e^* \\ 0 & Y & 0 & e^*\,(dk_2 + ek_1) \end{bmatrix} \tag{2.4}$$

where $X = b\eta^{\frac{1}{2}}a^* + (-ak_1 + ck_2)\,b^* + (ak_2 + ck_1)\,c^*$ *and* $Y = b^*\,(-dk_1 + ek_2) + c^*\,(dk_2 + ek_1)$

Proof. By using matrices (2.3) and (2.4), we can write

$$\begin{aligned}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}\begin{bmatrix} T_1^* \\ T_2^* \\ T_3^* \\ T_4^* \end{bmatrix}^{\mathbf{T}} &= \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 0 & 0 \\ d & 0 & e \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d* & 0 & e* \end{bmatrix}\begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}\right)^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 0 & 0 \\ d & 0 & e \end{bmatrix}\left(\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}\begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}^{\mathbf{T}}\right)\begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d* & 0 & e* \end{bmatrix}^{\mathbf{T}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ a & b & c \\ 1 & 0 & 0 \\ d & 0 & e \end{bmatrix}\left(\frac{1}{\eta^{\frac{1}{2}}}\begin{bmatrix} 0 & -k_1 & k_2 \\ \eta^{\frac{1}{2}} & 0 & 0 \\ 0 & k_2 & k_1 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & 0 \\ a^* & b^* & c^* \\ 1 & 0 & 0 \\ d* & 0 & e* \end{bmatrix}^{\mathbf{T}} \\
&= \frac{1}{\eta^{\frac{1}{2}}}\begin{bmatrix} 0 & -k_1 b^* + k_2 c^* & 0 & k_2 e^* \\ b\eta^{\frac{1}{2}} & X & b\eta^{\frac{1}{2}} & b\eta^{\frac{1}{2}} d^* + (ak_2 + ck_1)\, e^* \\ 0 & -k_1 b^* + k_2 c^* & 0 & k_2 e^* \\ 0 & Y & 0 & e^*\,(dk_2 + ek_1) \end{bmatrix}.
\end{aligned}$$

□

The position of the unit tangent vector field $T_1^*, T_2^*, T_3^*, T_4^*$ of ruled surfaces $\varphi_1^*, \varphi_2^*, \varphi_3^*, \varphi_4^*$, respectively, on the curve α^*, can be expressed by the bellowing matrix;

$$[T]\,[T^*]^{\mathbf{T}} = \begin{bmatrix} \langle T_1, T_1^* \rangle & \langle T_1, T_2^* \rangle & \langle T_1, T_3^* \rangle & \langle T_1, T_4^* \rangle \\ \langle T_2, T_1^* \rangle & \langle T_2, T_2^* \rangle & \langle T_2, T_3^* \rangle & \langle T_2, T_4^* \rangle \\ \langle T_3, T_1^* \rangle & \langle T_3, T_2^* \rangle & \langle T_3, T_3^* \rangle & \langle T_3, T_4^* \rangle \\ \langle T_4, T_1^* \rangle & \langle T_4, T_2^* \rangle & \langle T_4, T_3^* \rangle & \langle T_4, T_4^* \rangle \end{bmatrix}, \tag{2.5}$$

here $[T^*]^{\mathbf{T}}$ is the tranpose matrix of $[T^*]$.
The six pairs of Frenet ruled surface and involutive Frenet ruled surface have striction curves with orthogonal tangent vector fields, these are
Tangent and involutive tangent ruled surfaces of the α,
involutive binormal and tangent ruled surface of the α,
involutive tangent and binormal ruled surface of the α,
Binormal and involutive binormal ruled surfaces of the α,
Darboux and involutive tangent ruled surfaces of an α,
Darboux and involutive binormal ruled surfaces of an α.

Theorem 2.6. *Tangent vector fields of striction curves on tangent ruled surface and involutive normal ruled surface and binormal ruled surface have orthogonal under the condition are* $\dfrac{k_2}{k_1} = \dfrac{\left(\frac{k_1^*}{\eta^*}\right)' \eta^*}{k_1^* k_2^*}$.

Proof. Since the equations (2.4) and (2.5), we have

$$\langle T_1, T_2^* \rangle = \langle T_3, T_2^* \rangle = \frac{-k_1 b^* + k_2 c^*}{\eta^{\frac{1}{2}}} = 0 \Longrightarrow \frac{k_2}{k_1} = \frac{\left(\frac{k_1^*}{\eta^*}\right)' \eta^*}{k_1^* k_2^*},$$

this completes the proof. □

Theorem 2.7. *Tangent vector fields of striction curves on tangent ruled surface and binormal ruled surface and involutive Darboux ruled surface have orthogonal under the condition are* $\dfrac{k_2^{'}k_1 - k_1^{'}k_2}{\lambda k_1\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} = constant.$

Proof. From the equations (2.4) and (2.5), we have

$$\begin{aligned} \langle T_1, T_4^* \rangle &= \langle T_3, T_4^* \rangle = \frac{1}{\eta^{\frac{1}{2}}} k_2 e^* = 0 \Longrightarrow k_2 e^* = 0, k_2 \neq 0 \\ e^* &= 0 \Longrightarrow (\mu^*)' = 0 \Longrightarrow \frac{k_2^{'}k_1 - k_1^{'}k_2}{\lambda k_1\left(k_1^2+k_2^2\right)^{\frac{3}{2}}} = const., \end{aligned}$$

this completes the proof. □

Theorem 2.8. ***i)*** *Tangent vector fields of striction curves on normal and involutive tangent ruled surfaces have orthogonal under the condition are* $\left(\frac{k_1}{k_1^2+k_2^2}\right)' = 0.$
ii) *Tangent vector fields of striction curves on normal and involutive binormal ruled surfaces have orthogonal under the condition are* $\left(\frac{k_1}{k_1^2+k_2^2}\right)' = 0.$

Proof. **i)** By using the equations (2.4) and (2.5), we can write

$$\langle T_2, T_1^* \rangle = b = \frac{\left(\frac{k_1}{k_1^2+k_2^2}\right)'}{\|c_2'(s)\|} = 0 \Longrightarrow \left(\frac{k_1}{k_1^2+k_2^2}\right)' = 0,$$

this completes the proof.
ii) Since $\langle T_2, T_3^* \rangle = b$, it is trivial. □

Theorem 2.9. *Tangent vector fields of striction curves along normal and involutive normal ruled surfaces are orthogonal under the condition*

$$b\eta^{\frac{1}{2}} a^* + (-ak_1 + ck_2)\, b^* + (ak_2 + ck_1)\, c^* = 0.$$

Proof. Since the equations (2.4) and (2.5), we have

$$\langle T_2, T_2^* \rangle = \frac{X}{\eta^{\frac{1}{2}}} = 0 \Longrightarrow X = b\eta^{\frac{1}{2}} a^* + (-ak_1 + ck_2)\, b^* + (ak_2 + ck_1)\, c^* = 0,$$

this completes the proof. □

Theorem 2.10. *Tangent vector fields of striction curves along normal and involutive Darboux ruled surfaces are orthogonal under the condition*

$$b\eta^{\frac{1}{2}} d^* + (ak_2 + ck_1)\, e^* = 0.$$

Proof. Since $\langle T_2, T_4^* \rangle = \dfrac{b\eta^{\frac{1}{2}} d^* + (ak_2 + ck_1)\, e^*}{\eta^{\frac{1}{2}}}$ in the equations (2.4) and (2.5) and under the orthogonality condition $b\eta^{\frac{1}{2}} d^* + (ak_2 + ck_1)\, e^* = 0.$ □

Theorem 2.11. *Tangent vector fields of striction curves along Darboux ruled surface and involutive normal ruled surface are orthogonal under the condition*

$$\frac{k_1}{k_2} = \frac{(dc^* + eb^*)}{(db^* - ec^*)}.$$

Proof. Since the equations (2.4) and (2.5), we have

$$\begin{aligned} \langle T_4, T_2^* \rangle &= \frac{Y}{\eta^{\frac{1}{2}}} = 0 \Longrightarrow Y = b^* \left(-dk_1 + ek_2\right) + c^* \left(dk_2 + ek_1\right) = 0 \\ &\Longrightarrow \frac{k_1}{k_2} = \frac{(dc^* + eb^*)}{(db^* - ec^*)}, \end{aligned}$$

this completes the proof. □

Theorem 2.12. *Tangent vector fields of striction curves on involutive Darboux ruled surface and Darboux ruled surface are orthogonal under the condition* $(dk_2 + ek_1) = 0$ *or* $\left(\frac{k_2^*}{k_1^*}\right)' = const.$

Proof. By using the equations (2.4) and (2.5), we can write

$$\begin{aligned} \langle T_4, T_4^* \rangle &= \frac{e^* \left(dk_2 + ek_1\right)}{\eta^{\frac{1}{2}}} = 0 \Longrightarrow (dk_2 + ek_1) = 0 \;\; or e^* = 0 \\ e^* &= 0 \Longrightarrow \mu^* = const. \Longrightarrow \left(\frac{k_2^*}{k_1^*}\right)' = const. \end{aligned}$$

this completes the proof. □

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