PARTIAL DERIVATIVE EFFECTS IN TWO-DIMENSIONAL SPLINE FUNCTION NODES

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ABSTRACT. One of the methods is two-dimensional spline functions for to create geometrical model of surface. In this study Eligibility of partial derivatives values for each node was examined. These nodes are projection of creation aimed surface. Created effects by the chosen values were evaluated. The results of the application example was provided with a computer software developed.
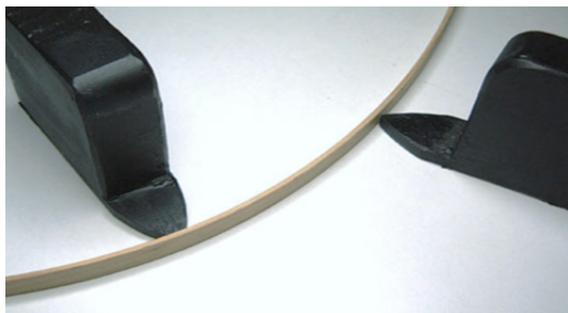
1. Introduction

FIGURE 1. Conversational usage of mechanical spline.

In mathematics, a spline is a numeric function that is piecewise-defined by polynomial functions([5][7]). In dictionary, the word "spline" originally meant a thin wood or metal slat in East Anglian dialect. By 1895 it had come to mean a flexible ruler used to draw curves[10]. These splines were used in the aircraft and ship-building industries. The successful design was then plotted on graph paper and the key points of the plot were re-plotted on larger graph paper to full size. The thin wooden strips provided an interpolation of the key points into smooth curves. The

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strips would be held in place at the key points (using lead weights called "ducks" or "dogs" or "rats")([6][7]) as shown in figure1. It is commonly accepted that the first mathematical reference to splines is the 1946 paper by [6], which is probably the first place that the word "spline" is used in connection with smooth, piecewise polynomial approximation([8][7]).

Let $T = (t_0, t_1, \cdots, t_{n-1})$ and $U = (u_0, u_1, \cdots, u_{n-1})$ here, $t_0 < t_1 < \cdots < t_{n-1}$ are distinct ordered real numbers and $u_0, u_1, \cdots, u_{n-1}$ are real numbers that represent each node. It describes a spline function f_{sp}

$$f_{sp}(t) = \begin{cases} f_0(t), & t_0 \leq t \leq t_1 \\ f_1(t), & t_1 < t \leq t_2 \\ \quad \vdots \\ f_{n-3}(t), & t_{n-3} \leq t \leq t_{n-2} \\ f_{n-2}(t), & t_{n-2} \leq t \leq t_{n-1} \end{cases}$$

$$f_j(t_j) = u_j, \qquad f_j(t_{j+1}) = u_{j+1}, \qquad j = 0, 1, \cdots, n-2.$$

$a, b \in R$, $a = t_0 < t_1 < \cdots < t_{n-2} < t_{n-1} = b$ is to be; $f_j : [t_j, t_{j+1}] \to R$, $j = 0, 1, \cdots, n-2$, $f_{sp} : [a, b] \to R$. Each f_j function may have any degree that is polynomial functions. Often the first, second and third order polynomial functions are used in practice([8][1]).
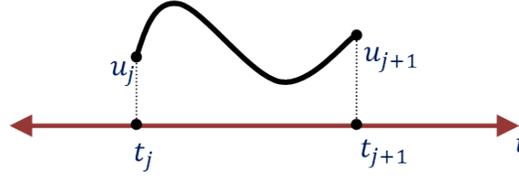

FIGURE 2. f_j piecewise function.

1.1. **Cubic spline functions.** Let $T = (t_0, t_1, \cdots, t_{n-1})$, $U = (u_0, u_1, \cdots, u_{n-1})$ and $G = (g_0, g_1, \cdots, g_{n-1})$. $f_{sp} : [t_0, t_{n-1}] \to R$, $u = f_{sp}(t)$, $t \in [t_0, t_{n-1}]$. $f_j : [t_j, t_{j+1}] \to R$, $f_j(t) = a_j t^3 + b_j t^2 + c_j t + d_j$, $j = 0, 1, \cdots, n-2$ which satisfied the conditions $f'_{sp}(t_i) = g_i$, $i = 0, 1, \cdots, n-1$ is unique [9].

$$f'_j(t_j) = g_j \text{ and } f_j(t_j) = u_j$$

$$f'_j(t_{j+1}) = g_{j+1} \text{ and } f_j(t_{j+1}) = u_{j+1}$$

$$j = 0, 1, \cdots, n-2$$

Condition can provides, at least third degree spline functions [9]. The cubic spline function $f_{sp}(t)$ has following representation [1].

$$w_i = \frac{1}{t_i - t_{i-1}} \left(\frac{u_i - u_{i-1}}{t_i - t_{i-1}} - g_{i-1} \right)$$

$$a_i = \frac{1}{t_i - t_{i-1}} \left(\frac{g_i - g_{i-1}}{t_i - t_{i-1}} - 2w_i \right)$$

$$b_i = -(t_i + 2t_{i-1}) a_i + w_i$$

$$c_i = g_{i-1} - 3a_i t_{i-1}^2 - 2b_i t_{i-1}$$

$$d_i = u_{i-1} - a_i t_{i-1}^3 - b_i t_{i-1}^2 - c_i t_{i-1}$$
$$i = 1, \ 2, \ \ldots, \ n-1$$

1.2. CubicSPL Cubic spline subroutine. The following subroutine representation have input values that are three vectors establish for cubic spline function and provision sought value of t. The result of this subroutine is a value that $u = f_{sp}(t)$.

   double CubicSPL (double* T, double* U, double* G, double t)

Example 1.1. T = $(1, \ 2, \ 3, \ 4, \ 5)$, U = $(-3, \ 3, \ 2, \ -2, \ 1)$ and G = $(0, \ 0, \ 0, \ 0, \ 0)$ are vectors representing the values of nodes.

```
#define TMax 5
T[TMax]  =  { 1, 2, 3, 4, 5 } ;
U[TMax]  =  { -3, 3, 2, -2, 1 } ;
G[TMax]  =  { 0, 0, 0, 0, 0 } ;
double t  =  3.7 ;
u  =  CubicSPL ( T, U, G, t ) ;
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$$u : -1.1359999999998536$$

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u  =  CubicSPL ( T, U, G, 2.07 ) ;
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$$u : 2.9859860000000111$$

Graphical representation of the results are also observed at figure 3.
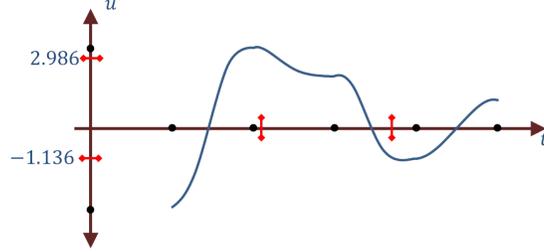

FIGURE 3. Graphical representation of example 1.1.

2. TWO DIMENSIONAL SPLINE

$a, b, c, d \in R$ and $\Omega = [a, b] \times [c, d]$, consider the rectangle on tOx plane as Ω region.

$$a = t_0 \ < \ t_1 \ < \ \cdots \ < t_i \ < \ \cdots \ < t_{m-1} = b; \ m \geq 1$$
$$c = x_0 \ < \ x_1 \ < \ \cdots \ < x_j \ < \ \cdots \ < x_{n-1} = d; \ n \geq 1$$
$$i = 0, \ 1, \ \cdots, \ m-1, \ j = 0, \ 1, \ \cdots, \ n-1$$

Ω region divided into $(n-1) \times (m-1)$ sub regions.

$$\Omega_{i,j} = \{(t, x) : \ t_i \ \leq \ t \ \leq \ t_{i+1}, \ x_j \ \leq \ x \ \leq \ x_{j+1}\}$$

$i = 0, \ 1, \ \cdots, \ m-2; \ j = 0, \ 1, \ \cdots, \ n-2$. For any $\Omega_{i,j}$ sub region have this edge cardinal points:

$$\zeta_{t_i, \ x_j}, \zeta_{t_{i+1}, \ x_j}, \ \zeta_{t_{i+1}, \ x_{j+1}}, \ \zeta_{t_i, \ x_{j+1}}$$

The cardinal points of each $\Omega_{i,j}$ sub region defines a grid Ω_{grd}. Be introduced a function $\lambda : \Omega_{grd} \to R$, $\lambda(t_i, x_j) = u_{(i,j)}$ on the grid extended on the Ω region [8].

$$U = \left\{ u_{(0,0)}, \ u_{(0,1)}, \ \cdots, \ u_{(0,n-1)}, \ u_{(1,0)}, \ \cdots, \ u_{(m-1,n-1)} \right\}$$

$$G_t = \left\{ g_{t(0,0)}, \ g_{t(0,1)}, \ \cdots, \ g_{t(0,n-1)}, \ g_{t(1,0)}, \ \cdots, \ g_{t(m-1,n-1)} \right\}$$

$$G_x = \left\{ g_{x(0,0)}, \ g_{x(0,1)}, \ \cdots, \ g_{x(0,n-1)}, \ g_{x(1,0)}, \ \cdots, \ g_{x(m-1,n-1)} \right\}$$

$$u_{(i,j)} \in R, \ g_{t(i,j)} \in R, \ g_{x(i,j)} \in R$$

$$\lambda(t_i, x_j) = u_{(i,j)}, \ \lambda'_t(t_i, x_j) = g_{t(i,j)}, \ \lambda'_x(t_i, x_j) = g_{x(i,j)},$$

$$f : \Omega \to R, \ f(t_i, x_j) = u_{(i,j)}, \ \lambda(t_i, x_j) = f(t_i, x_j)$$

$$i = 0, \ 1, \ \cdots, \ m-1, \ j = 0, \ 1, \ \cdots, \ n-1$$

The purpose is find $f : \Omega \to R$, $f(t,x)$ derivable real function [8].

$$H(t_0, x), \ H(t_1, x), \ H(t_2, x), \ \ldots, \ H(t_{m-1}, x), \qquad x_0 \ \leq \ x \ \leq \ x_{m-1}$$

$$S(t, x_0), \ S(t, x_1), \ S(t, x_2), \ \ldots, \ S(t, x_{n-1}), \qquad t_0 \ \leq \ t \ \leq \ t_{n-1}$$

$H(t_i, x)$, $i = 0, \ 1, \ \cdots, \ m-1$, $x_0 \ \leq \ x \ \leq \ x_{n-1}$ describe direction of x spline functions and $S(t, x_j)$, $j = 0, \ 1, \ \cdots, \ n-1$, $t_0 \ \leq \ t \ \leq \ t_{m-1}$ describe direction of t spline functions[8].

U, G_x and G_t data sets according with Ω_{grd}. These sets provides m amounts $U_{\overline{\overline{X}}_i} = \left\{ u_{(i,j)} \mid j = 0, \ 1, \ \cdots, \ n-1 \right\}$ and $G_{\overline{\overline{X}}_i} = \left\{ g_{x(i,j)} \mid j = 0, \ 1, \ \cdots, \ n-1 \right\}$ vectors for each $H(t_i, x)$ spline functions direction of x and n amounts $U_{\overline{\overline{T}}_j} = \left\{ u_{(i,j)} \mid i = 0, \ 1, \ \cdots, \ m-1 \right\}$ and $G_{\overline{\overline{T}}_j} = \left\{ g_{t(i,j)} \mid i = 0, \ 1, \ \cdots, \ m-1 \right\}$ vectors for each $S(t, x_j)$ spline functions direction of t. At the end of the $m + n$ amounts supply one-dimensional spline functions can be calculated.
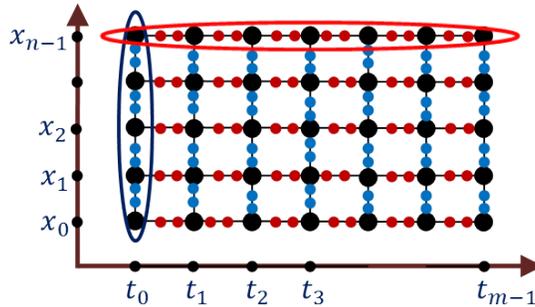

FIGURE 4. $m + n$ amounts one-dimensional spline functions.

FIGURE 5. The demonstration will consist of an auxiliary spline function according to the direction.

3. ANY $f(t,x)$ ON THE Ω

Calculations can be started with the any direction spline functions the direction of t or direction of x arbitrarily chosen. Let $t_0 \leq l \leq t_{m-1}$ and $x_0 \leq k \leq x_{n-1}$. If t direction spline functions are chosen, a supplementary spline function can create using these spline functions. The solution is shown below.

Let $k \in (x_0, x_{n-1})$ and $l \in (t_0, t_{m-1})$. $u_{(t_{sup}, j)} = S(l, x_j)$, $j = 0, 1, \cdots, n-1$, $f(l, k) = H(t_{sup}, k)$. In detail $u_{(t_{sup}, j)} = CubicSPL(\text{T}, \text{U}_{\overline{\overline{T}}_j}, \text{G}_{\overline{\overline{T}}_j}, l)$; for $j = 0, 1, \cdots, n-1$ create a new $\text{U}_{\overline{\overline{X}}_{sup}}$ vector for use in x direction. Therefor $CubicSPL$ function need a $\text{G}_{\overline{\overline{X}}_{sup}}$ vector represent x direction derivative values of $H(t_{sup}, x)$ $t_i \leq l \leq t_{i+1}$, $\text{G}_{\overline{\overline{X}}_i}$ and $\text{G}_{\overline{\overline{X}}_{i+1}}$ vectors represent partial derivative values relationship $H(t_i, x)$ and $H(t_{i+1}, x)$ spline functions on direction x. Get help these two vectors to determine $\text{G}_{\overline{\overline{X}}_{sup}}$. $\text{U}_{\overline{\overline{X}}_{sup}}$ was obtained. $t_i \leq l \leq t_{i+1}$ and $j = 0, 1, \cdots, n-1$. As shown in figure 6.

FIGURE 6

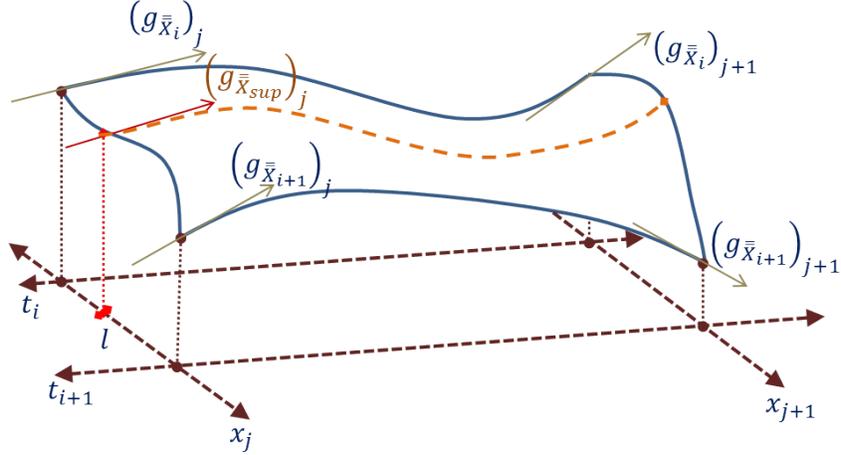$$\left(g_{\overline{\overline{X}}_{sup}}\right)_j = \left(g_{\overline{\overline{X}}_i}\right)_j \frac{|t_{i+1} - l|}{|t_{i+1} - t_i|} + \left(g_{\overline{\overline{X}}_{i+1}}\right)_j \frac{|t_i - l|}{|t_{i+1} - t_i|}$$

$$f(l, k) = CubicSPL(\text{X}, \text{U}_{\overline{\overline{X}}_{sup}}, \text{G}_{\overline{\overline{X}}_{sup}}, k);$$

4. Smooth Surface

At the direction of t and the direction of x, partial derivative values can be arbitrarily chosen on the grid nodes. Nevertheless the created surface able to reach somewhat smoothness using some basic rules. For spline functions direction of t :

$$\left(g_{\overline{\overline{T}}_j}\right)_0 = \frac{\left(u_{\overline{\overline{T}}_j}\right)_1 - \left(u_{\overline{\overline{T}}_j}\right)_0}{t_1 - t_0}$$

$$\left(g_{\overline{\overline{T}}_j}\right)_{m-1} = \frac{\left(u_{\overline{\overline{T}}_j}\right)_{m-2} - \left(u_{\overline{\overline{T}}_j}\right)_{m-1}}{t_{m-2} - t_{m-1}}$$

$$\left(g_{\overline{\overline{T}}_j}\right)_i = \left(\frac{\left(u_{\overline{\overline{T}}_j}\right)_i - \left(u_{\overline{\overline{T}}_j}\right)_{i-1}}{t_i - t_{i-1}} \frac{|t_{i+1} - t_i|}{|t_{i+1} - t_{i-1}|} + \frac{\left(u_{\overline{\overline{T}}_j}\right)_{i+1} - \left(u_{\overline{\overline{T}}_j}\right)_i}{t_{i+1} - t_i} \frac{|t_{i-1} - t_i|}{|t_{i+1} - t_{i-1}|} \right)$$

$$i = 1, \ 2, \ \cdots, \ m-2, \ j = 0, \ 1, \ \cdots, n-1.$$

For spline functions direction of x :

$$\left(g_{\overline{\overline{X}}_i}\right)_0 = \frac{\left(u_{\overline{\overline{X}}_i}\right)_1 - \left(u_{\overline{\overline{X}}_i}\right)_0}{x_1 - x_0}$$

$$\left(g_{\overline{\overline{X}}_i}\right)_{n-1} = \frac{\left(u_{\overline{\overline{X}}_i}\right)_{n-2} - \left(u_{\overline{\overline{X}}_i}\right)_{n-1}}{x_{n-2} - x_{n-1}}$$

$$\left(g_{\overline{\overline{X}}_i}\right)_j = \left(\frac{\left(u_{\overline{\overline{X}}_i}\right)_j - \left(u_{\overline{\overline{X}}_i}\right)_{j-1}}{x_j - x_{j-1}} \frac{|x_{j+1} - x_j|}{|x_{j+1} - x_{j-1}|} + \frac{\left(u_{\overline{\overline{X}}_i}\right)_{j+1} - \left(u_{\overline{\overline{X}}_i}\right)_j}{x_{j+1} - x_j} \frac{|x_{j-1} - x_j|}{|x_{j+1} - x_{j-1}|} \right)$$

$$i = 0, \ 1, \cdots, m-1, \ j = 1, 2, \ \cdots, n-2.$$

5. Results and Discussion

A computer program was developed as a result of this study is. Using the *http://oguzersinan.net.tr* web address that is accessible to this computer program.

$$U = \begin{pmatrix} 3 & 4 & 3 \\ 4 & 5 & 4 \\ 3 & 3 & 3 \end{pmatrix}, \quad G_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \quad G_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ get in that}$$

way. Surface appearance is shown in figure 7. Computer software by the method described hereinabove, when it determines partial derivatives of nodes is calculated

as $G_x = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ and $G_t = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{pmatrix}$. New surface appearance

is shown in figure 7.

Determine the value of partial derivatives with the weighted arithmetic mean method on two-dimensional cubic spline functions reveals appropriate results.
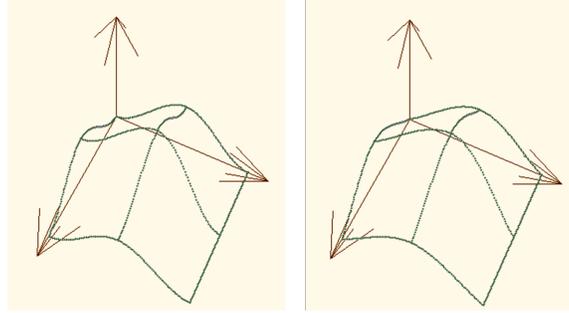
FIGURE 7. On left side without correction, on right side after smoothness correction.

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