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ON HADAMARD-TYPE INEQUALITIES FOR k-FRACTIONAL INTEGRALS

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Abstract. In this paper we prove Hadamard-type inequalities for k-fractional Riemann-Liouville integrals and Hadamard-type inequalities for fractional Riemann-Liouville integrals are deduced. Also we deduced some well known results related to Hadamard inequality.

1. introduction

Fractional Calculus is a branch of mathematical study that developed from the established definitions of calculus integral and derived operators [2].

Fractional calculus was mainly a study kept for the finest minds in mathematics. Fourier, Euler, Laplace are among those mathematicians who showed a casual interest by fractional calculus and mathematical consequences. A lot of them established definitions by means of their own notion and style. Most renowned of these definitions are the Grunwald-Letnikov and Riemann-Liouville definition [4].

There are many types of fractional integrals have been defined in literature, the most classical are Riemann-Liouville fractional integrals defined as follows:

Definition 1.1. Let $f \in L_1[a,b]$, then Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as:

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a \tag{1.1}$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b. \tag{1.2}$$

For further details one may see [3, 6, 7].

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[1] If $k > 0$, then k-Gamma function Γ_k is defined as:

$$\Gamma_k(\alpha) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{\alpha}{k}} - 1}{(\alpha)_{n,k}}.$$

If $\Re(\alpha) > 0$ then k-Gamma function in integral form is defined as

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt,$$

with the property that

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha)$$

In [5] k-fractional Riemann-Liouville integrals are defined as follows:

Let $f \in L_1[a, b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a \tag{1.3}$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b. \tag{1.4}$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals.

Besides applications of fractional integrals in applied sciences, now a days many researchers in the field of pure mathematics, for example mathematical analysis have studied them extensively see [2, 3, 4, 6].

In [8], Sarikaya et al. proved the following Hadamard-type inequalities for Riemann-Liouville fractional integrals.

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on [a,b], then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{(\frac{a+b}{2})+} f(b) + I^\alpha_{(\frac{a+b}{2})-} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.5}$$

with $\alpha > 0$.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}[I^\alpha_{(\frac{a+b}{2})+} f(b) + I^\alpha_{(\frac{a+b}{2})-} f(a)] - f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left[((\alpha+1)|f'(a)|^q + (\alpha+3)|f'(b)|^q)^{\frac{1}{q}}\right. \\ &\left. + ((\alpha+3)|f'(a)|^q + (\alpha+1)|f'(b)|^q)^{\frac{1}{q}}\right]. \end{aligned} \tag{1.6}$$

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q > 1$, then the following inequality for fractional* *integral holds:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}[I^{\alpha}_{(\frac{a+b}{2})+}f(b)+I^{\alpha}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \tag{1.7}$$
$$\leq \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In this paper we generalize the fractional Hadamard-type inequalities (1.5), (1.6) and (1.7) via k-fractional integrals and show that these inequalities are special cases of our results. Also we deduced some well known results.

2. Hadamard-type inequalities for k-fractional integrals

Here we give k-fractional Hadamard-type inequalities.

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for k-fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{2.1}$$

with $\alpha,k>0$.

Proof. From convexity of f we have

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2}. \tag{2.2}$$

Putting $x=\frac{t}{2}a+\frac{(2-t)}{2}b, y=\frac{(2-t)}{2}a+\frac{t}{2}b$ for $t\in[0,1]$. Then $x,y\in[a,b]$ and above equation gives

$$2f\left(\frac{a+b}{2}\right)\leq f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right), \tag{2.3}$$

multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$, and integrating over $[0,1]$ we have

$$\frac{2k}{\alpha}f\left(\frac{a+b}{2}\right)\int_0^1 t^{\frac{\alpha}{k}-1}dt$$
$$\leq\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt$$
$$=\frac{2^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)\right],$$

from which one can have

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)\right]. \tag{2.4}$$

On the other hand convexity of f gives

$$f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\leq\frac{t}{2}f(a)+\frac{2-t}{2}f(b)+\frac{2-t}{2}f(a)+\frac{t}{2}f(b),$$

multiplying both sides of above inequality with $t^{\frac{\alpha}{k}-1}$, and integrating over $[0,1]$ we have

$$\begin{aligned}&\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{\frac{\alpha}{k}-1}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\\ &\leq[f(a)+f(b)]\int_0^1 t^{\frac{\alpha}{k}-1}dt,\end{aligned}$$

from which one can have

$$\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)\right]\leq\frac{f(a)+f(b)}{2}. \tag{2.5}$$

Combining inequality (2.4) and inequality (2.5) we get inequality (2.1) . □

Remark 2.1. If we take $k=1$, Theorem 2.1 gives inequality (1.5) of Theorem 1.1 and putting $\alpha=1$ along with $k=1$ in Theorem 2.1 we get the classical Hadamard inequality.

3. k-fractional inequalities related to Hadamard inequality

For next results we need the following lemma.

Lemma 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f^{'}\in L[a,b]$, then the following equality for k-fractional integrals holds:*

$$\begin{aligned}&\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\\ &=\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1 t^{\frac{\alpha}{k}}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right].\end{aligned} \tag{3.1}$$

Proof. One can note that

$$\begin{aligned}&\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right]\\ &=\frac{b-a}{4}\left[t^{\frac{\alpha}{k}}\frac{2}{a-b}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)|_0^1-\int_0^1\frac{\alpha}{k}t^{\frac{\alpha}{k}-1}\frac{2}{a-b}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]\\ &=\frac{b-a}{4}\left[-\frac{2}{b-a}f\left(\frac{a+b}{2}\right)-\frac{2\alpha}{k(a-b)}\int_b^{\frac{a+b}{2}}\left(\frac{2}{b-a}(b-x)\right)^{\frac{\alpha}{k}-1}\frac{2}{a-b}f(x)dx\right]\end{aligned}$$

(3.2)

$$=\frac{b-a}{4}\left[-\frac{2}{b-a}f\left(\frac{a+b}{2}\right)+\frac{2^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}+1}}I^{\alpha,k}_{(\frac{a+b}{2})-}f(b)\right].$$

Similarly

$$
\begin{aligned}
&-\frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}} f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right] \\
(3.3)\qquad &= -\frac{b-a}{4}\left[\frac{2}{b-a}f\left(\frac{a+b}{2}\right)-\frac{2^{\frac{\alpha}{k}+1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}+1}} I^{\alpha,k}_{(\frac{a+b}{2})+}f(a)\right].
\end{aligned}
$$

Combining (3.2) and (3.3) one can have (3.1). □

Using the above lemma we give the following k-fractional Hadamard-type inequality.

Theorem 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q\geq 1$, then the following inequality for k-fractional integrals holds:*

$$
\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\
(3.4)\qquad &\leq \frac{b-a}{4(\frac{\alpha}{k}+1)}\left(\frac{1}{2(\frac{\alpha}{k}+2)}\right)^{\frac{1}{q}}\left[\left(\left(\left(\frac{\alpha}{k}+1\right)\right)|f'(a)|^q+\left(\frac{\alpha}{k}+3\right)|f'(b)|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left(\frac{\alpha}{k}+3\right)|f'(a)|^q+\left(\frac{\alpha}{k}+1\right)|f'(b)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

with $\alpha, k>0$.

Proof. From Lemma 3.1 and convexity of $|f'|$ and for $q=1$ we have

$$
\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4}\int_0^1 t^{\frac{\alpha}{k}}\left(\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right)dt. \\
&= \frac{b-a}{4\left(\frac{\alpha}{k}+1\right)}[|f'(a)|+|f'(b)|].
\end{aligned}
$$

For $q>1$ we proceed as follows. Using Lemma (3.1) we have

$$
\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].
\end{aligned}
$$

Using power mean inequality we get

$$
\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{\frac{\alpha}{k}+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].
\end{aligned}
$$

Convexity of $|f'|^q$ gives

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{\frac{\alpha}{k}+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{\frac{\alpha}{k}}\left(\frac{t}{2}|f'(a)|^q+\frac{2-t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\int_0^1 t^{\frac{\alpha}{k}}\left(\frac{2-t}{2}|f'(a)|^q+\frac{t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right] \\
&= \frac{b-a}{4}\left(\frac{1}{\frac{\alpha}{k}+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f'(a)|^q}{2(\frac{\alpha}{k}+2)}+\frac{|f'(b)|^q}{\frac{\alpha}{k}+1}-\frac{|f'(b)|^q}{2(\frac{\alpha}{k}+2)}\right]^{\frac{1}{q}}+\left[\frac{|f'(a)|^q}{\frac{\alpha}{k}+1}-\frac{|f'(a)|^q}{2(\frac{\alpha}{k}+2)}\right.\right. \\
&\left.\left.+\frac{|f'(b)|^q}{2(\frac{\alpha}{k}+2)}\right]^{\frac{1}{q}}\right],
\end{aligned}$$

which after a little computation gives the required result. □

Remark 3.1. If we take $k=1$ in Theorem 3.1, we get inequality (1.6) of Theorem 1.2 and if we take $\alpha=q=1$ along with $k=1$ in Theorem 3.1, then inequality (3.4) gives inequality the following result.

Corollary 3.1. *With assumptions of Theorem 3.1 we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right).$$

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for $q>1$, then the following inequality for k-fractional integral holds:*

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right] \qquad (3.5)\\
&\leq \frac{b-a}{4}\left(\frac{4}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|],
\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 3.1 we have

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4}\left[\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\int_0^1 t^{\frac{\alpha}{k}}\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].
\end{aligned}$$

From $H\ddot{o}lder's$ inequality we get

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\left[\left[\int_0^1 t^{\frac{\alpha p}{k}}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{\frac{\alpha p}{k}}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right]^{\frac{1}{q}}\right].$$

Convexity of $|f'|^q$ gives

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\left(\frac{1}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1\left(\frac{t}{2}|f'(a)|^q+\frac{2-t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1\left(\frac{2-t}{2}|f'(a)|^q+\frac{t}{2}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{b-a}{4}\left(\frac{1}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right]^{\frac{1}{q}}\right].$$

For second inequality of (3.5) we use Minkowski's inequality as

$$\left|\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}[I^{\alpha,k}_{(\frac{a+b}{2})+}f(b)+I^{\alpha,k}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{16}\left(\frac{4}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}\left[[|f'(a)|^q+3|f'(b)|^q]^{\frac{1}{q}}+[3|f'(a)|^q+|f'(b)|^q]^{\frac{1}{q}}\right]$$
$$\leq\frac{b-a}{16}\left(\frac{4}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}(3^{\frac{1}{q}}+1)(|f'(a)|+|f'(b)|)$$
$$\leq\frac{b-a}{16}\left(\frac{4}{\frac{\alpha p}{k}+1}\right)^{\frac{1}{p}}4(|f'(a)|+|f'(b)|).$$

□

Remark 3.2. For $k=1$ in above theorem we get inequality (1.7). If we take $\alpha=k=1$ we get the following result.

Corollary 3.2. *With assumptions of Theorem 3.2 we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[(|f'(a)|^q+3|f'(b)|^q)^{\frac{1}{q}}+(3|f'(a)|^q+|f'(b)|^q)^{\frac{1}{q}}\right].$$

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